MAC-CPTM Situations Project

Situation 49: Similarity

Prompt

In a geometry class, students were given the diagram in Figure 1 depicting two acute triangles, ΔABC and $\Delta A'B'C'$, and students were told that $\Delta ABC \sim \Delta A'B'C'$ with a figure (Figure 1) indicating that $A'B' = 2AB$ and $m\angle B = 75°$. From this, a student concluded that $m\angle B' = 150°$.

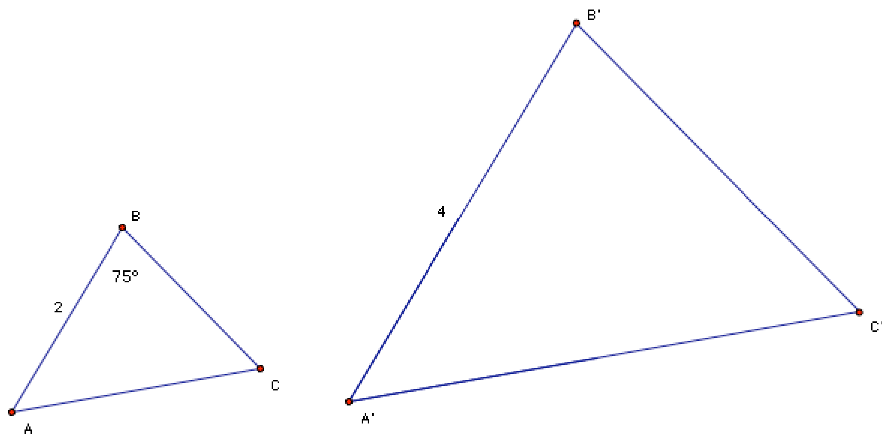

Figure 1.

Commentary[1]

By definition, two polygons are similar if and only if their corresponding angles are congruent and their corresponding side lengths are proportional. Thus, similar figures may have different sizes, but they have the same shape. The following foci incorporate a variety of approaches (geometric, graphical, and symbolic) to shed light on the concept of similarity. The first focus refutes the claim made in the Prompt by appealing to the definition of similar triangles, and the second focus refutes the claim using an indirect proof that considers the impact of doubling the measures of each of the angles of the original triangle. In Foci 3 and 4, similarity is examined in terms of transformations in general, and dilations in particular. Under a geometric similarity transformation, angle measure is preserved and the ratio of the measures of corresponding distances is constant. [Finally, a geometric construction and proof is provided which lends further insight into the definition of similarity.] In each focus, the concept of ratio is emphasized, because common ratio lies at the heart of why size, but not shape, may change for similar figures.

[1] Each of these foci assumes a context of Euclidean geometry.

Mathematical Foci

Mathematical Focus 1

Length measures of corresponding parts of similar triangles are related proportionally with a constant of proportionality that may be other than 1, but angle measures of corresponding angles of similar triangles are equal.[2]

By definition, similar triangles have corresponding angles that are congruent and corresponding sides that are proportional. Figure 1 in the prompt depicts two similar triangles, ΔABC and $\Delta A'B'C'$, with $AB = 2$ and $A'B' = 4$. A ratio of the lengths of the sides AB and $A'B'$ can be used to determine the corresponding lengths of the sides of $\Delta A'B'C'$ as scaled sides of ΔABC or vice versa. In particular, since $A'B = 2AB$ and ΔABC and $\Delta A'B'C'$ are similar, it must be true that $A'C' = 2AC$, and $B'C' = 2BC$. Although the constant of proportionality, 2, can be used to find lengths of corresponding sides of these similar triangles, it does not apply to the measures of the angles, since the angles must, by definition, be congruent.

Mathematical Focus 2

The degree measures of a triangle cannot be some non-unit multiple of the corresponding degree measures in a similar triangle.

Suppose that one doubled the degree measures of each of the angles of ΔABC. This would result in the sum of the degree measures of the angles of $\Delta A'B'C'$ being 360. But this is not possible because the sum of the degree measures of the angles of any triangle is 180. So, the degree measures of the angles A', B', and C' cannot each be double the corresponding degree measure in ΔABC.

[2] Consideration of other similar figures is beyond the scope of this focus.

More generally, consider ΔABC ~ ΔA′B′C′ with $A'B' = k * AB$ where k>0 and k≠1. If one supposed that each degree measure in ΔA′B′C′ were k times the corresponding degree measure in ΔABC, then the sum of the degree measures of the angles of ΔA′B′C′ would be 180k. This would contradict the fact that the sum of the degree measures of a triangle must be 180. Therefore, the degree measures of the angles of ΔA′B′C′ cannot be k times the degree measures of the corresponding angles of ΔABC, where k≠1.

Mathematical Focus 3
Similar triangles have corresponding angles that are congruent and have corresponding sides whose lengths are proportional. Given that either one of these properties is true, the other must be true.

Each of the two conditions for triangle similarity—congruence of corresponding angles of two triangles and proportionality among their corresponding sides—can be shown to imply the other. So, if the lengths of the sides of the similar triangles in the prompt are proportional, then the corresponding angles must be congruent.

First, it can be shown that, given that the three angles of one triangle are congruent to the three angles of another triangle, the lengths of the corresponding sides of the two triangles are proportional. This is Proposition 4 in Book VI of Euclid's *Elements* (Densmore, 2002).

Consider triangles ΔABC and ΔDEF such that $\angle A \cong \angle D, \angle B \cong \angle E, \text{and } \angle C \cong \angle F$ (see Figure 2). Map ΔDEF to ΔDE'F' about D so that $\overline{E'F'} \parallel \overline{BC}$.

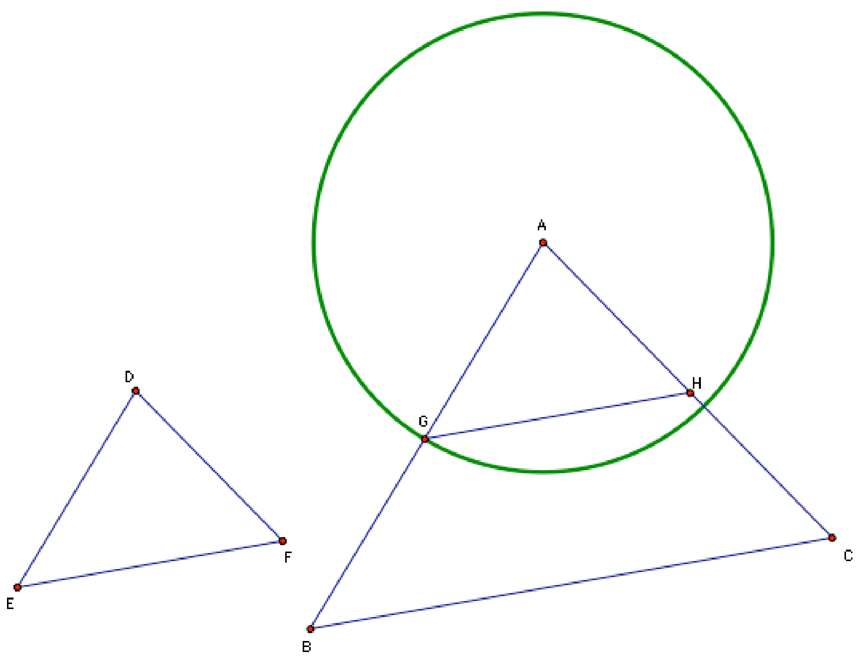

Figure 2.

Construct a circle centered at A with radius $\overline{DE}$. Let G be the point of intersection of the circle and $\overline{AB}$. Construct a line through G parallel to $\overline{BC}$. Let H be the point of intersection of the parallel line and $\overline{AC}$. Since $\overleftrightarrow{GH} \parallel \overleftrightarrow{BC}$, corresponding angles are congruent, so ∠AGH ≅ ∠E. Therefore, ΔAGH ≅ ΔDEF by the ASA theorem of congruence (∠AGH ≅ ∠E' ≅ ∠E, $\overline{AG} \cong \overline{DE'} \cong \overline{DE}$ and $\angle A \cong \angle D$). Since $\overleftrightarrow{GH} \parallel \overleftrightarrow{BC}$, and parallel lines divide transversals proportionally, $\frac{GB}{AG} = \frac{HC}{AH} \Rightarrow \frac{AB}{AG} = \frac{AC}{AH} \Rightarrow \frac{AB}{AC} = \frac{AG}{AH}$, which is equivalent to $\frac{AB}{AG} = \frac{AC}{AH}$. Since AG = DE and AH = DF, by substitution, $\frac{AB}{DE} = \frac{AC}{DF}$. Using similar reasoning, it can be shown that $\frac{BC}{EF} = \frac{AB}{DE}$. Therefore, by the transitive property of equality: $\frac{BC}{EF} = \frac{AC}{AH}$. Thus, it can be concluded that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

Conversely, given that the lengths of the corresponding sides of two triangles are proportional, it can be shown that the angles of one triangle are congruent to the corresponding angles of the other triangle. This is Proposition 5 in Book VI of Euclid's *Elements*.

Consider triangles ΔABC and ΔDEF such that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ as illustrated in Figure 3.

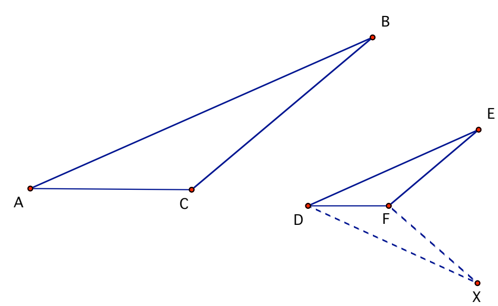

Figure 3.

At point D on $\overrightarrow{DF}$ copy *∠CAB in the half-plane of* $\overleftrightarrow{DF}$ *in which point E does not lie*, and at point F on $\overrightarrow{FD}$ copy *∠ACB in the half-plane of* $\overleftrightarrow{DF}$ *in which point E does not lie*. Let X be the point of intersection of the two non-concurrent rays of the copied angles. (Note: A more complete proof might also establish that point X exists, namely, that the non-concurrent rays of the two copied angles must intersect.) Because two angles of Δ*ABC* are congruent to two angles of ΔDXF, the remaining angles, $\angle$B and $\angle$X are congruent. By the preceding proof (Euclid's Proposition 4), since the angles of ΔABC and ΔDXF are congruent, the lengths of the corresponding sides of ΔABC and ΔDXF are proportional. So, $\frac{AB}{AC} = \frac{DX}{DF}$ and from the given proportion, $\frac{AB}{AC} = \frac{DE}{DF}$. So, $\frac{DE}{DF} = \frac{DX}{DF}$, implying that $DE = DX$. Similarly, $EF = XF$, since $\frac{BC}{XF} = \frac{AC}{DF}$ from the proportional sides of ΔABC and Δ*DXF*, and $\frac{BC}{EF} = \frac{AC}{DF}$ from the given proportion. Since $\overline{DF}$ is a side of both Δ*DEF* and Δ*DXF*, $\Delta DXF \cong \Delta DEF$ by the SSS triangle congruence postulate. So, $\angle X \cong \angle E$, $\angle EDF \cong \angle XDF$, and $\angle EFD \cong \angle XFD$. Since the angles of ΔABC and the corresponding angles of Δ*DXF* were already established as being congruent, and since the corresponding angles of ΔDXF and Δ*DEF* are congruent, by transitivity of congruence the corresponding angles of ΔABC and Δ*DEF* are congruent.

Mathematical Focus 4

When comparing corresponding parts of triangles and their images under similarity transformations, angle measure is preserved and the length of a side of the image triangle is the product of the ratio of similitude and the length of the corresponding side of the original triangle. Isometries, for which the ratio of similitude is 1, are a subset of similarity transformations.

Transformations in which shape is preserved but size is not necessarily preserved are similarity transformations. Given ΔABC, consider Δ A′B′C′ to be the image of a similarity transformation of ΔABC. A geometric similarity transformation is an angle-preserving function such that all distances are scaled by a constant ratio, $k \neq 0$. For the given similarity transformation, the lengths of sides of ΔA′B′C′ are double the length of the corresponding sides of ΔABC, and, because angles are preserved by the similarity transformation, the angles of ΔA′B′C′ are congruent to the corresponding angles of ΔABC.

Dilations whose center is the origin in a coordinate plane are similarity transformations of the form $F((x, y)) = (kx, ky)$ for some constant ratio $k \neq 0$. In general, if $|k| < 1$, the mapping is a contraction, for which the resulting image is smaller than its pre-image. If $|k| > 1$, the mapping results in an expansion, for which the resulting image is larger than its pre-image. If $k = 1$, the mapping is the identity transformation under composition of transformations, for which the resulting image is the same size and shape as its pre-image. If $k = -1$, size and shape are also preserved.

Transformations in which shape and size are preserved are known as isometries, for which the resulting image is congruent to the pre-image. There are five distinct types of isometries: identity, reflection, non-identity rotation, non-identity translation, and glide reflection. In the coordinate plane, any figure may be mapped to a similar figure by a composition of dilations and isometries. The constant of proportionality is the product of the ratios of similarity. When one figure is mapped to its image by only isometries, the product of the ratios is a power of 1 (which is equal to 1) and the figures are congruent.

Mathematical Focus 5

For a triangle inscribed in a circle and its dilation through the center of the circle, the relationship between the inscribed angle and the length of the intercepted arc shows that angle measure is preserved under dilation.

Using Geometer's Sketchpad, a dynamic diagram (see Figure 4) can be created to illustrate that shape and angle measure are preserved for similar triangles for which one can be represented as the expansion or contraction of the other triangle (with the center of the circle that inscribes that triangle as the center of dilation). Consider a triangle inscribed in a circle centered at the origin. Using polar coordinates, the coordinates of any point on the circle are (r,θ), where r is the radius of the circle and θ is the measure of the angle in standard position formed by the x-axis and a ray from the origin to a point on the circle. For any angle in the triangle, the measure of the angle is equal to half the length of the intercepted arc divided by the radius. Thus, angle measure is a function of arc length and radius, namely, $m(a,r)=\frac{a}{2r}$. Since the length of an arc is equal to the product of the radian measure of the arc angle and the length of radius of the circle, the ratio of half the length of the arc to the radius of the circle will be equal to the radian measure of the angle in the triangle intercepting that arc, regardless of the length of the radius. Therefore, as a circle centered at the origin is expanded or contracted, the measure of any angle in the inscribed triangle remains constant. Thus, angle measure is preserved.

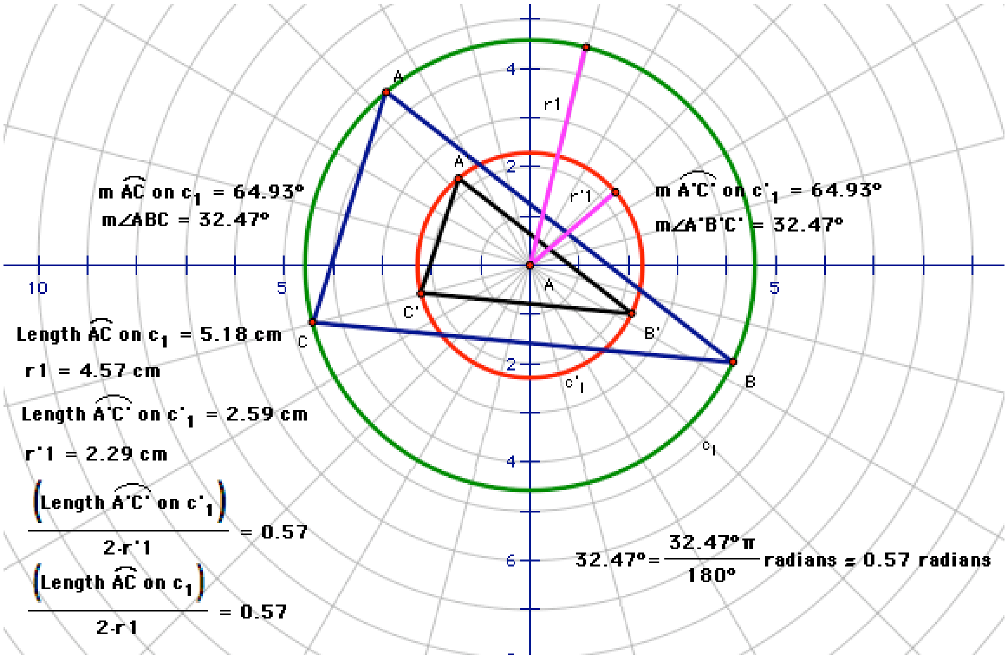

Figure 4.

Reference

Densmore, D. (Ed.). (2002). *Euclid's elements* (The Thomas L. Heath Translation). Santa Fe, NM: Green Lion Press.